COLLECTIONELSS MACHINE LEARNING THE HAMILTONIAN FRAMEWORK

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OUTLINE

1. The path of Artificial Intelligence Part I

- **2. A paradigm-shift: The connectionist wave**
- **3. Collectionless AI**
- **4. Intelligence and laws of Nature**

5. The Hamiltonian framework of learning Part II

- **6. Cognidynamics: A Theory of Neural propagation**
- **7. Neural vs wave propagation**

LEARNING IN THE TEMPORAL DIMENSION *J*[0*,T*](*w*) = *J^T* + *dt L*(*x*(*t*)*, w*(*t*)*, t*)*,* (2)

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A HJB equations **VALUE FUNCTION**

Value Function $V : [0, T] \times \mathcal{X} \to \mathbb{R} : (t, \xi) \mapsto V(t, x)$

$$
V(t,x):=J_T+\min_w\int_t^T ds\ L(\xi(s),w(s),s)
$$

USING BELLMAN' PRINCIPLE USING BELLMAN' PRINCIPLE

$$
V(t, x^*) = V(t + \Delta t, x^* + \Delta x^*) + \min_{w([t, t + \Delta t])} L(x(t), w(t), t) \Delta t + o(\Delta t)
$$

= $V(t, x^*) + V_s(t, x^*) \Delta t + V_x(t, x^*) \Delta x^* + o(\Delta x^*) + o(\Delta t)$
+ $\min_{w([t, t + \Delta t])} L(x(t), w(t), t) \Delta t,$

w([*t,t*+*t*]) $\mathsf{R3P.2024}$. Hamiltonian learning as well as the corresponding function which return it at time *t*. S3P-2024 - Hamiltonian Learning

HJB EQUATIONS ciple we get

$$
V(t, x^*) = V(t + \Delta t, x^* + \Delta x^*) + \min_{w([t, t + \Delta t])} L(x(t), w(t), t) \Delta t + o(\Delta t)
$$

= $V(t, x^*) + V_s(t, x^*) \Delta t + V_x(t, x^*) \Delta x^* + o(\Delta x^*) + o(\Delta t)$
+ $\min_{w([t, t + \Delta t])} L(x(t), w(t), t) \Delta t,$
 $\dot{x}(t) \Delta t = f(x^*, w^*, t) \Delta t$

$$
o(\Delta t) = V_x(t, x^*) \cdot f(x^*, w^*, t) \Delta t + V_s(t, x^*(t)) \Delta t + \min_{w([t, t + \Delta t])} L(x(t), w(t), t) \Delta t
$$

$$
V_s(t, x^*) = -\min_{\omega} \left(L(x^*, \omega, t) + V_x(t, x^*) \cdot f(x^*, \omega, t) \right)
$$

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HAMILTONIAN AND HJB EQUATIONS *^o*(*t*) = *^Vx*(*t, x*?)*·f*(*x*?*, w*?*, t*)*^t* ⁺ *^Vs*(*t, x*?(*t*))*^t* + min *tion defined by eq. (120) with the terminal boundary condition* ⁸*^x* ² ^R*ⁿ* : *^V* (*T,x*) = HAMILTONIAN AND HJB EQUATIONS

$$
H(x, p, s) := \min_{\omega} \left(L(x, \omega, s) + p \cdot f(x, \omega, s) \right) \quad \text{Hamiltonian}
$$
\n
$$
V_s(t, x^*) = -\min_{\omega} \left(L(x^*, \omega, t) + V_x(t, x^*) \cdot f(x^*, \omega, t) \right)
$$
\n
$$
\sum_{\substack{\text{all } \text{all} \\ \text{all } \\ V(T, x) = g(x) \quad \text{terminal condition}
$$
\n
$$
V(x, x) = g(x) \quad \text{terminal condition}
$$

on *V* (*t, x*) which can be solved under the respect of the terminal condition

 α

HJ(B) EQUATIONS AND METHOD OF CHARACTERISTICS \blacksquare njej EQUATIONS AND tions for time-independent Hamiltonians also works for the general case of time- $H/(R)$ foi iations and $\frac{1}{2}$ for conception $\frac{1}{2}$ for the understanding schemes $\frac{1}{2}$ WE I HOD OF CHARAC I ERISTICS

Hamiltonian dynamics is sufficient Let us consider the following (HJ) initial-point problem

(HJ)
$$
\begin{cases} V_s(t,x) + H(x, V_x(t,x,t)) = 0, \\ V(0,x) = g(x). \end{cases}
$$

 \sim \sim \sim

 W_{α} want to convert this PDF problem into \mathcal{W}_{α} that can enour a dram let us the value of considerational perspective W_0 and the method of characteristic and che comp **p** θ /*t* We want to convert this PDE problem into an ODE that can open a dramatically different computational perspective. We use the method of characteristic. Now, let us introduce the *co-state p* as $p : \mathbf{R} \times \mathbb{R}$ and consider the total derivative¹⁸ of its κ coordinate $V(0, x) = g(x)$.

PDE problem into an ODE that can

be respective. We use the method of tate p as p : $\frac{1}{2}\sqrt{x}$ and consider the

(*t*) := ˙*p^x* (*t*) = *^V^x^t*(*t, x*(*t*)) + *^V^xxⁱ · ^x*˙ *ⁱ.* (163)

HJ(B) EQUATIONS AND METHOD OF CHARACTERISTICS $HJ(B)$ EQUATIONS AND *V* (*V* (*D*) = *g*(*x*)*.* (102) WE and the *internet* to an *o*pen and *internet* can open a dramatic can open a dramat *Vs*(*t, x*) + *H*(*x, Vx*(*t, x, t*)=0*. V* (2008) EQUATIONS AND METHOD OF CHARACTERISTICS

 $co\text{-}state\,\,p\,\,\text{as}\,\,p:=V_x$

 \Box erent deer it erelye? *p* ρ *p* and *p* and *p* and *p* and *c p* and *p* and *p* and *c p* a dierent computational perspective. We use the method of characteristic. Now, we use the method How does it evolve?

 $\dot{p}^{\kappa}(t) := \dot{p}_{x_{\kappa}}(t) = V_{x_{\kappa}t}(t, x(t)) + V_{x_{\kappa}x_i} \cdot \dot{x}_i.$

Now, if *V* solves (HJ) then Now, if *V* solves (HJ) then $\frac{1}{2}$

$$
V_{x_{\kappa}t}(x,t) = -H_{x_{\kappa}}(x, V_x(x,t),t) - H_{p_i}(x, V_x(x,t),t) \cdot V_{x_i x_{\kappa}}(x,t)
$$

$$
\dot{p}^{\kappa}(t) = -H_{x_{\kappa}}(x(t), \underbrace{V_x(x(t), t)}_{p(t)}, t) + (\dot{x}_i(t) - H_{p_i}(x(t), \underbrace{V_x(x(t), t)}_{p(t)}, t)) \cdot V_{x_{\kappa}x_i}(t, x(t))
$$

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HJB EQUATIONS AND METHOD OF CHARACTERISTICS (CON'T) Now, if *V* solves (HJ) then *V^x^t*(*x, t*) = *H^x* (*x, Vx*(*x, t*)*, t*) *H^pⁱ* (*x, Vx*(*x, t*)*, t*) *· V^xix* (*x, t*)*. {H, H}* = 0. We have that the null total derivative *d* OF CHARACTERISTICS (CON'T)

$$
\dot{p}^{\kappa}(t) = -H_{x_{\kappa}}(x(t), \underbrace{V_x(x(t), t)}_{p(t)}, t) + (\dot{x}_i(t) - H_{p_i}(x(t), \underbrace{V_x(x(t), t)}_{p(t)}, t)) \cdot V_{x_{\kappa}x_i}(t, x(t))
$$

Now we can promptly see that the following choice where we will show that classic Hamiltonian dynamics that satisfies that satisfies that satisfies the HJB equality of \mathcal{L}

(H)

$$
\begin{cases}\n\dot{x}(t) = H_p(x(t), p(t), t) \\
\dot{p}(t) = -H_x(x(t), p(t), t)\n\end{cases}
$$
\n(HJ)

$$
\begin{cases}\nV_s(t, x) + H(x, V_x(t, x, t) = 0. \\
V(0, x) = g(x).\n\end{cases}
$$

= *H*(*x*(*t*)*, p*(*t*)*, t*) + *p*(*t*) *· Hp*(*x*(*t*)*, p*(*t*)*, t*)*.* its coordinate S3P-2024 - Hamiltonian Learning

NON-HOLONOMIC CONSTRAINTS

$$
J_{[0,T]}(w) = J_T + \int_0^T dt \ L(x(t), w(t), t)
$$

$$
\dot{x}(t) = f(x(t), w(t), t)
$$

S3P-2024 - Hamiltonian Learning a stochastic process, when the stochastic process, whereas *we can* in diagnosis in a paraise *we can u* and *u* and actions of two players the task of which is the task of \sim

\mathbf{A} possible way to attack the the optimization of (2) is the constraint (1) is in the cons LAGRANGIAN APPROACH A possible way to attack the the the optimization of (2) under the constraint (1) is constrai LAGRANGIAN APPROACH

$$
J_L = J_T + \int_0^T dt \bigg(L(x(t), w(t), t) + \lambda(t) \cdot (f(x(t), w(t), t)) - \dot{x}(t) \bigg)
$$

 $\mathcal{U}\left(x(t), \lambda(t), y(t), t\right) := L\left(x(t), y(t), t\right) + \lambda(t), f(x(t), y(t), t)\right)$ $\mathcal{H}\big(x(t),\lambda(t),w(t),t\big):=L\big(x(t),w(t),t\big)+\lambda(t)\cdot f(x(t),w(t),t)\big)$

$$
J_L = J_T + \int_0^T dt \bigg(\underbrace{\mathcal{H}(x(t), \lambda(t), w(t), t) - \lambda(t) \cdot \dot{x}(t)}_{\mathcal{L}^x} \bigg)
$$

S3P-2024 - Hamiltonian Learning *dt* (*t*) *· ^x*˙(*t*) = (*t*) *· x*(*t*) $\overline{}$ ia \overline{a} μ *n*iltonian Learning

A CLASSIC "TRICK" \overline{a} classic (*t*)*x*^(*t*)) *J^L* = *J^T* + $CLASSIC "TRICK"$

$$
\int_0^T dt \ \lambda(t) \cdot \dot{x}(t) = \left[\lambda(t) \cdot x(t) \right]_0^T - \int_0^T dt \ \dot{\lambda}(t) \cdot x(t)
$$
\n
$$
J_L = J_T + \int_0^T dt \left(\underbrace{\mathcal{H}(x(t), \lambda(t), w(t), t) - \lambda(t) \cdot \dot{x}(t)}_{\mathcal{L}^x} \right) \ \underbrace{\underbrace{\mathcal{L}^x}_{\text{Syst}}}_{\text{Syst}} \tag{5.12}
$$
\n
$$
J_L(x, \lambda) = J_T - \left[x(t) \cdot \lambda(t) \right]_0^T + \int_0^T dt \left(\underbrace{\mathcal{H}(x(t), \lambda(t), w(t), t) + x(t) \cdot \dot{\lambda}(t)}_{\mathcal{L}^{\lambda}} \right) \ \underbrace{\left(\sum_{t=1}^{T} \lambda(t) \cdot \lambda(t) \right)}_{\text{Syst}} \ \underbrace{\mathcal{L}^x}_{\text{Syst}} \tag{5.22}
$$

S3P-2024 - Hamiltonian Learning *^w*˙ *^L ^w* ! *Hx*(*x*(*t*)*,* (*t*)*, w*(*t*)*, t*)=0*.*

EULER-LAGRANGE EQUATIONS use both its representations given by eq. (133) and eq. (134). We get $\mathsf{R} \mathcal{F}$ *d* $EULER$ LAGRANGE EQUATIONS

$$
0 = \frac{d}{dt}\mathcal{L}_x^x - \mathcal{L}_x^x \to \dot{\lambda}(t) + \mathcal{H}_x(x(t), \lambda(t), w(t), t) = 0
$$

\n
$$
0 = \frac{d}{dt}\mathcal{L}_\lambda^\lambda - \mathcal{L}_\lambda^\lambda \to \dot{x}(t) - \mathcal{H}_\lambda(x(t), \lambda(t), w(t), t) = 0
$$

\n
$$
0 = \frac{d}{dt}\mathcal{L}_w^\lambda - \mathcal{L}_w^\lambda \to \mathcal{H}_x(x(t), \lambda(t), w(t), t) = 0.
$$

$$
H(x,\lambda,t)=\min_w {\cal H}(x,\lambda,w,t).
$$

 $\lim_{n \to \infty}$ Finally, this leads to the Hamiltonian equations

$$
\begin{cases}\n\dot{\lambda}(t) = -H_x(x(t), \lambda(t), t) \\
\dot{x}(t) = -\mathcal{H}_{\lambda}(x(t), \lambda(t), t).\n\end{cases}
$$

x
x^{(*x*}) *x S3P-2024 - Hamiltonian Learning*

CHARACTERISTIC EQUATIONS OF HJB Theorem 1. *Let us assume that the learning of agent L can be framed as the* **HAMILTONIAN "LAWS"** HAMILTONIAN "I AWS" \sim *minimization of R*[0*,T*]*. The optimal policy w*? = arg min *R*[0*,T*](*w*) *is determined*

$$
\begin{cases}\n\dot{x}(t) &= H_p(x(t), p(t), u(t), t) = f(x(t), w(t), u(t)) \\
\dot{p}(t) &= -H_x(x(t), p(t), u(t), t),\n\end{cases}
$$

$$
x(0) = x_0
$$
 and
$$
p(T) = p_T = V_x(T, x(T))
$$

CLASSIC CASE OF LINEAR QUADRATIC (LQ) CONTROL

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LINEAR QUADRATIC (LQ) CONTROL $A = \sum_{i=1}^n A_i B_i$ equation of the following let us consider the followi $PATIC$ M *RWADRA* \mathbf{r}

$$
\begin{aligned}\n\dot{x} &= Ax + Bw \\
L(x, w, t) &= \frac{1}{2} x' Qx + \frac{1}{2} w' Rw \\
w^* &= \min_w \left(\frac{1}{2} x' Qx + \frac{1}{2} w' Rw + p' (Ax + Bw) \right) \\
&= -R^{-1} B' p = -R^{-1} B' P x := Fx.\n\end{aligned}
$$

2

Fundack for *p* such that *p* Furthermore we look for *p* such that *p* = *P x*. Now the optimal control strategy $A + DI$ $f_{\alpha, \alpha}$ denote the dynamics of system (48) into $f_{\alpha, \alpha}$ into f_{α} $\dot{x} = (A + BF)x$. feedback control

The control strategy based on *w*? acts as a feedback control by the state and

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◆

THE HAMILTONIAN

$$
w^* = \min_w \left(\frac{1}{2} x' Q x + \frac{1}{2} w' R w + p' (Ax + Bw) \right)
$$

= $-R^{-1} B' p = -R^{-1} B' P x := F x.$

$$
H(x, p, w)|_{w^*} = \frac{1}{2}x'Qx + \left[\frac{1}{2}w'Rw + p' \cdot (Ax + Bw)\right]_{w^*}
$$

= $\frac{1}{2}x'Qx + \frac{1}{2}(R^{-1}B'p)'R(R^{-1}B'p) - p' \cdot (Ax + B(R^{-1}B'p))$
= $\frac{1}{2}x'Qx + \frac{1}{2}p'\underbrace{BR^{-1}B'}_{S}p - p' \cdot (Ax + BR^{-1}B'p)$
= $\frac{1}{2}x'Qx - \frac{1}{2}p'Sp + p'Ax$

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SOLVING HJB EQUATIONS *S* $\overline{}$ \mathbf{c} *<u>PQUATIONS</u>* SOLVING HJB EQUATIONS **SOLVING HJB EQUATIONS**

The control strategy based on *w*? acts as a feedback control by the state and $V(t, x) = \frac{1}{2}$ 2 $x'P(t)x$ (Let's assume) a quadratic function The control strategy based on *w*? acts as a feedback control by the state and $P(f(x)) = \frac{1}{2}P(f)$ *^P*¯)+(*A*⁰ *PBR* ¯ ¹*B*⁰)*P*¯ \overline{z} Let's assume a quadratic function

 $V_t + H(x, V_x) = 0$ terminal condition $V(T, x) = g(x)$ $V \perp H(a, V) = 0$ denoted positive $V(T, x) =$ \mathbf{v}_t **1** $\mathbf{u}(\omega, \mathbf{v}_x) = 0$ communion $\mathbf{v}(\mathbf{v}, \omega) = 0$ $g(x) = g(x)$ $\left(\begin{array}{ccc} P & P & P \\ P & P & Q \end{array} \right)$ such a given and P

$$
\frac{1}{2}x'\dot{P}x + \frac{1}{2}x'Qx - \frac{1}{2}p'Sp + p'Ax = \frac{1}{2}x'\dot{P}x + \frac{1}{2}x'Qx - \frac{1}{2}x'P'SPx + x'P'Ax = 0
$$

 $a_{\kappa i} \leadsto PA$ ⁼ *R*¹*B*⁰ *^p* ⁼ *R*¹*B*⁰ ⁷ Using tensorial notation we have *A*⁰ $P \rightsquigarrow a_{\kappa i} p_{\kappa j} = p_{j\kappa} a_{\kappa i} \rightsquigarrow PA.$

 $\dot{P} + Q + PA + A'P - PSP = 0.$ If *P* is symmetric we have⁷ *A*⁰ *P* = *P A* and since the above equation holds for Riccati equation

r-2024 - Hamiltonia
P ^x˙ ⁼ *Ax* ⁺ *Bw* ⁼ *Ax BR*¹*^B* S3P-2024 - Hamiltonian Learning

SOLVING HJB EQUATIONS *x*˙ = (*A* + *BF*)*x.* \overline{AB} **EOUATIONS** and
And

$$
V_t + H(x, V_x) = 0 \t terminal condition \t V(T, x) = g(x)
$$

\nWe solve
\n
$$
\overrightarrow{P} + Q + PA + A'P - PSP = 0 \t V(t, x) = \frac{1}{2}x'P(t)x
$$

\nWe find the control law solution of the PDE
\n
$$
w^* = \min_w \left(\frac{1}{2}x'Qx + \frac{1}{2}w'Rw + p'(Ax + Bw) \right)
$$

\n
$$
= -R^{-1}B'p = -R^{-1}B'Px := Fx.
$$

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ASYMPTOTIC STABILITY *P*˙ + *Q* + *P A* + *A*⁰ *P PSP* = 0*.* (58) The boundary condition involves the value *P*(*T*) that is given. Basically ¹ ASYMPIONIC STABILITY P **PD APP 27 APP for any** α *P*˙+ *Q* + *P A* + *A*⁰

$$
\begin{aligned}\n\dot{x} &= Ax + Bw \\
\dot{x} &= (A + BF)x \qquad \text{feedback control} \\
W(t) &= \frac{1}{2}x'(t)\bar{P}x(t) \qquad \text{Lyapunov function} \\
\dot{W}(t) &= \dot{x}'\bar{P}x(t) = x'\bar{P}(A - BR^{-1}B'\bar{P})x(t) \\
&= \frac{1}{2}x'(t)\Big(\bar{P}(A - BR^{-1}B'\bar{P}) + (A' - \bar{P}BR^{-1}B')\bar{P}\Big)x(t)\n\end{aligned}
$$

$$
\bar{P}(A - BR^{-1}B'\bar{P}) + (A' - \bar{P}BR^{-1}B')\bar{P}
$$
\n
$$
= \bar{P}A + A'\bar{P} - 2\bar{P}S\bar{P} = \underbrace{Q + \bar{P}A + A'\bar{P} - \bar{P}S\bar{P}}_{0} - Q - \bar{P}S\bar{P} = -Q - \bar{P}S\bar{P}
$$

P PSP = 0*.* (52)

P aip^j = *pjaⁱ P A*. *^x*˙ ⁼ *Ax* ⁺ *Bw* ⁼ *Ax BR*¹*^B* S3P-2024 - Hamiltonian Learning

◆

ASYMPTOTIC STABILITY (CON'T) *^W*(*t*) = ¹ $\mathbf l$ *x*0 (*t*)*P x*¯ (*t*)*,* ASYMPTOTIC STABILITY (CON'T)

$$
\dot{W}(t) = \dot{x}' \bar{P}x(t) = x' \bar{P}(A - BR^{-1}B'\bar{P})x(t)
$$

= $\frac{1}{2}x'(t)\left(\bar{P}(A - BR^{-1}B'\bar{P}) + (A' - \bar{P}BR^{-1}B')\bar{P}\right)x(t)$

$$
\bar{P}(A - BR^{-1}B'\bar{P}) + (A' - \bar{P}BR^{-1}B')\bar{P}
$$
\n
$$
= \bar{P}A + A'\bar{P} - 2\bar{P}S\bar{P} = Q + \bar{P}A + A'\bar{P} - \bar{P}S\bar{P} - Q - \bar{P}S\bar{P} = -Q - \bar{P}S\bar{P}
$$
\n
$$
\hat{W}(t) = -\frac{1}{2}x'(Q + \bar{P}S\bar{P})x \le 0
$$
\n
$$
Q + \bar{P}S\bar{P} \ge 0 \qquad Q > 0, R > 0.
$$
\nasymptotic stability

\nThe "magic" of asymptotic stability we need to solve Riccat's equation

LQ: HAMILTONIAN EQUATIONS $\overline{2}$ **P assumed that** P **is symmetric and positive definition** \bigcap . **H**

$$
w^* = \min_w \left(\frac{1}{2} x' Q x + \frac{1}{2} w' R w + p' (Ax + Bw) \right)
$$

= $-R^{-1} B' p = -R^{-1} B' P x := F x.$

 $\dot{x} = (A + BF)x$ This makes it possible to determine the Hamiltonian control to determine the Hamiltonian control to determine t
This makes it possible to determine the Hamiltonian control to determine the Hamiltonian control to determine

$$
H(x, p, w)|_{w^*} = \frac{1}{2}x'Qx + \left[\frac{1}{2}w'Rw + p' \cdot (Ax + Bw)\right]_{w^*}
$$

= $\frac{1}{2}x'Qx + \frac{1}{2}(R^{-1}B'p)'R(R^{-1}B'p) - p' \cdot (Ax + B(R^{-1}B'p))$
= $\frac{1}{2}x'Qx + \frac{1}{2}p'\underline{BR^{-1}B'}p - p' \cdot (Ax + BR^{-1}B'p)$
= $\frac{1}{2}x'Qx - \frac{1}{2}p'Sp + p'Ax$

 $t = 0.05$ into dynamics of system (69) into dynamics of \sim *x*˙ = (*A* + *BF*)*x.* (73) S3P-2024 - Hamiltonian Learning

LQ HAMILTONIAN EQUATIONS LQ HAMILTONIAN EQUATIONS

$$
\begin{aligned}\n\dot{x} &= Ax + Bw = Ax - \underbrace{BR^{-1}B}_{S}p \\
\dot{p} &= -Qx - A'p.\n\end{aligned}\n\qquad\n\begin{pmatrix}\n\dot{x} \\
\dot{p}\n\end{pmatrix}\n=\n\begin{pmatrix}\nA & -S \\
-Q & -A'\n\end{pmatrix}\n\cdot\n\begin{pmatrix}\nx \\
p\n\end{pmatrix}
$$

This can be compact that the compact \mathcal{L}_max

1-Dim

$$
\begin{pmatrix} a & -s \\ -q & -a \end{pmatrix} \qquad \det \begin{pmatrix} \rho - a & -s \\ -q & \rho + a \end{pmatrix} = 0
$$

positive eigenvalues ... we need of the crystal ball!

$$
(\rho^2 - a^2) - qs = 0 \rightarrow \rho = \pm \sqrt{a^2 + qs}
$$

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HAMILTON and RICCATI EQUATIONS *x p* 2 **ATI FO** *Q A*⁰ *· x p* ˙ ✓ *x p* $\overline{}$ \overline{M} and R *a*¹¹ $\overline{}$ *·* ✓ *x p* UAMILTON and BICCAT

When considering the circuital assumption $p = Px$ we get $\dot{p} = \dot{P}x + P\dot{x}$. From the state equation $P\dot{x} = PAx - PSPx$ and, therefore, When considering the circuital assumption $p = Px$ we get $\dot{p} = P$. the state equation $P x = P A x = P B T x$ *p* = ✓ *^A ^S* $\frac{1}{2}$ *A p* \dot{r} [When considering the circuital assumption *p* = *P x* we get ˙*p* = *P x*˙ + *Px*˙. From

$$
\dot{p} = -Qx - A'Px = \dot{P}x + PAx - PSPx
$$

Phat is, for any x *:*

$$
Qx + A'Px + \dot{P}x + PAx - PSPx = 0 \to \dot{P} + Q + A'P + PA - PSP = 0.
$$

 $\dot{P} + Q + A'P + PA - PSP = 0$

 $P + PA - PSP = 0$ it cannot be solved "forward in time"!

HAND HJB EQUATIONS: CAN WE FIND THE VALUE FUNCTION? WHY IS QUADRATIC? CAN WE FIND THE VALUE FUNCTION WHY IS OUADRATIC equations. According to the HJB equations. CAN WE FIND THE VALUE FUNCTION? WHY IS OUADRATIC? **is the SND HIB EQUATIONS:** of Riccati's equation. We can also easily see that the quadratic assumption on CAN WE FIND THE VALUE FUNCTION! WHY IS QU CAN WE FIND THE VALL ALUE FUNCTION? WHY IS QUADRATIC?

$$
V_t + H(x, V_x) = V_t + \frac{1}{2}x'Qx - \frac{1}{2}p'Sp + p'Ax = 0
$$
 Hamiltonian
\n
$$
\dot{x} = Ax + Bw = Ax - \underbrace{BR^{-1}Bp}_{S}
$$
\n
$$
\dot{p} = -Qx - A'p.
$$
\n
$$
p'\dot{x} - x'\dot{p} = p'Ax - p'Sp + x'Qx + x'A'p \leftarrow \begin{cases} p'\dot{x} = p'Ax - p'Sp \\ x'\dot{p} = -x'Qx - x'A'p \end{cases}
$$

$$
V_t = \tfrac{1}{2} \big(p' \dot{x} - x' \dot{p} \big)
$$

V^t + \overline{c} *x*^(*x*) *<i>p*(*x*) *x*(*t*) *xx xx x* the state $p(t) = P(x)x(t)$ we get \overline{y} + \overline{y} $x(t)$ $P(\ell) - I(x) \mathcal{X}(\ell)$ we get *Px*˙ *x*⁰ . Since $p(t) = P(x)x(t)$ we get $p(t) = P(x)x(t)$ we get

$$
V_t + \frac{1}{2} \left(x' P \dot{x} - x' (\dot{P} x + P \dot{x}) \right) = V_t - \frac{1}{2} x' \dot{P}(t) x' = 0
$$

and, finally

and, finally
\n
$$
V(t,x) = \frac{1}{2}x'(t)\int_0^t ds \dot{P}(s)x(t) = \frac{1}{2}x'(t)P(t)x(t)
$$

S3P-2024 - Hamiltonian Learning can be determined from Riccati equation (78) by initializing *P*(*T*) = *P^T* which Because of the meaning of the value function we can also conclude that *P* 0. It S3P-2024 - Hamiltonian Learning can be determined from Riccati equation (78) by initializing *P*(*T*) = *P^T* which

TO SUM UP

- HJB: necessary and SUFFICIENT conditions!
- H equations are characteristic for the HJ PDE
- Links with Lagrangian approach - [Pontryagin's](https://en.wikipedia.org/wiki/Pontryagin%27s_maximum_principle) Maximum Principle - PMP)
- The perspective of H Learning

COGNIDYNAMICS: A THEORY OF NEURAL PROPAGATION

"Life can only be understood backwards; but it must be lived forwards."

Søren Kierkegaard

LEARNING IN RECURRENT NETS *µ* = \$*µ*⇠*µ,* ' \$'⇠'*,* ! \$!⇠!*,* \$ ⇠ (156) **IFARNING IN RECLIRRENT NETS** $\overline{}$ is will be in remainder of the paper, the paper, the paper, the paper, the paper, the paper of crucial importance for $\overline{}$ LEARNING IN RECURRENT NETS.

$$
\mathcal{N}: \begin{cases} \xi_i(t) = u_i(t) & i \in \mathcal{I} \\ \dot{\xi}_i(t) = \alpha_i(t) \left[-\xi_i(t) + \sigma \left(\sum_{j \in \mathcal{V}} w_{ij}(t) \xi_j(t) \right) \right] & i \in \bar{\mathcal{V}} \\ w_{ij}(t) = \psi_{ij}(t) \nu_{ij}(t) & (i,j) \in \mathcal{A} \\ w_{ij}(t) = \omega_{ij}(t) w_{ij}(t) & (i,j) \in \mathcal{A} \end{cases}
$$

$$
x \sim [\xi_i, w_{ip}]
$$

environmental interaction

$$
R(\nu, T) = \int_0^T \left(\frac{1}{2} \sum_{i \in \bar{\mathcal{V}}} \sum_{j \in \mathcal{V}} \frac{m_{ij}(s) \nu_{ij}^2(s) + \gamma \sum_{i \in \mathcal{O}} \phi_i(s) V(\xi_i(s), s) \right) ds
$$

tional sense. We only have a graph of connections with weights ✓*ip*. As we can see $S3P-ZUZ4$ - (Neural propagation • The kinetic energy term *K*(*s*) = ¹ ²*mij* (*s*)⌫² *ij* (*s*) of the particle (*i, j*) that is S3P-2024 - Neural propagation

THE HAMILTONIAN

$$
H(\xi_i, w_{ij}, p_i, p_{ij}, t)
$$
\n
$$
= \min_{v} \left(\frac{1}{2} \sum_{ij} m_{ij}(t) \nu_{ij}^2(t) + \gamma \phi_i(t) V(\xi_i, t) \right)
$$
\n
$$
+ \sum_{i} \alpha_i(t) p_i(t) \left[-\xi_i(t) + \sigma \left(\sum_{j} \omega_{ij}(t) w_{ij}(t) \xi_j(t) \right) \right]
$$
\n
$$
+ \sum_{ij} p_{ij}(t) \psi_{ij}(t) \nu_{ij}(t)
$$
\n
$$
+ \beta_{ij} \left(\sum_{j} \psi_{ij}(t) \psi_{ij}(t) \right)
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+ \beta_{ij} \left(\sum_{j} \psi_{ij}(t) \psi_{ij}(t) \right)
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$$
+ \beta_{ij} \left(\sum_{j} \psi_{ij}(t) \right)
$$

$$
H(\xi_i, w_{ij}, p_i, p_{ij}, t)
$$
\n
$$
= \min_{v} \left(\frac{1}{2} \sum_{i} m_{ij}(t) \nu_{ij}^2(t) + \gamma \phi_i(t) V(\xi_i, t) \right)
$$
\n
$$
= \min_{v} \left(\frac{1}{2} \sum_{i} m_{ij}(t) \nu_{ij}^2(t) + \gamma \phi_i(t) V(\xi_i, t) \right)
$$

 ⇠*i*(*t*) + enta !*ij* (*t*)*wij* (*t*)⇠*^j* (*t*) ⇣X developmental learning

ij (*t*)*.*

$$
H = -\frac{1}{2} \sum_{ij \in \mathcal{A}} \beta_{ij}(t) p_{ij}^2(t) + \gamma \sum_{i \in \mathcal{O}} \phi_i(t) V(\xi(t), t)
$$

$$
+ \sum_{i \in \bar{\mathcal{V}}} \alpha_i(t) p_i(t) \Big[-\xi_i(t) + \sigma \Big(\sum_{j \in \mathcal{V}} \omega_{ij}(t) w_{ij}(t) \xi_j(t) \Big) \Big]
$$

 $S3P-20$ ↵*i*(*t*)*pi*(*t*) ⇠*i*(*t*) + *is those variables are given but must respect asymptotic convergence. On the The optimal solution of oracle O*² *only needs to respect conditions (6), that* S3P-2024 - Neural propagation *opposite, in order to yield dissipation, the causal solution involves all variables.*

NIAN (CON'T) **THE HAMILTONIAN (CON'T)**

H

 $\overline{}$ $\overline{}$ $\frac{1}{2}$

 $\,+\,$

 $\sum_{i\in$

 $\alpha_i(t) p_i(t) \left[\begin{array}{c} 0 \end{array} \right]$

 $\overline{}$

 $\boldsymbol{\xi}_i(t) +$

9 $\overline{}$

 $\sum_{j\in$

(4)

V ${\cal E}$ *ij* (*t*) \mathcal{C} *ij* (*t*) ⇠*j*

V¯

 \blacktriangledown

 ϖ *ij*

 $d(\mathcal{H})$

ij 2

(*t*) +

 \prec

 $\sum_{i\in$

 $\phi_i(t) V$

 $\overline{}$

 $\xi(t), t$

O

ij ጠ *A*

S3P-2024 - Neural propagation $F3D3034$ by the Hamiltonian of the system. is driven in a new framework of optimization in the enriched space of weights (⌫*,* ⇣). l
ti *terms,*

full characterization of learning full characterization of learning

LEARNING IN THE TEMPORAL DIMENSION *proposed related formulation of learning involves a related* cognitive action *that must be minimized.*

 $F3D2024$ Nouvel propagation S3P-2024 - Neural propagation

SST-2024 - IN FORTWARD WAS ARRESTED ON A TEN- $331 - 2021 - 18$ calcut at propagation S3P-2024 - Neural propagation

GRADIENT-BASED INTERPRETATION OF HAMILTONIAN LEARNING COADIENIT DACED INITEDDDETATIONL **UNADILIS PUASED INTENTNE FATION** *i j* \overline{a} *ij* (*t*) 2*mij* (*t*) \overline{P} *i* ↵*i*(*t*)*pi*(*t*) *ij* (*t*) = *ij* (0) *·* exp Lemma 2. *The dissipation parameter ij evolves according to* Z *^t* ✓*ij* (*s*)*ds* (13) ✓*ij* := *m*˙ *ij* ˙ *mij* **N** LE *ij*

$$
\theta_{ij} := \frac{\dot{m}_{ij}}{m_{ij}} - 2\frac{\dot{\psi}_{ij}}{\psi_{ij}} \qquad \beta_{ij}(t) := \frac{\psi_{ij}^2(t)}{m_{ij}(t)}
$$

$$
\dot{\beta}_{ij} = \frac{d}{dt} \frac{\psi_{ij}^2}{m_{ij}} = \frac{\psi_{ij}^2}{m_{ij}} \left(2\frac{\dot{\psi}_{ij}}{\psi_{ij}} - \frac{\dot{m}_{ij}}{m_{ij}} \right) = -\frac{\psi_{ij}^2}{m_{ij}} \theta_{ij} = -\beta_{ij} \theta_{ij}
$$

$$
\beta_{ij}(t) = \beta_{ij}(0) \cdot \exp\left(-\int_0^t \theta_{ij}(s)ds\right)
$$

$$
\overbrace{\dot{w}_{ij}(t)} = -\beta_{ij}(t)\overbrace{\dot{p}_{ij}(t)}^{\text{I-gradient-based interpretation}}
$$

S3P-2024 - Neural propagation **i**s S3P-2024 - Neural propagatio \overline{a}

GRADIENT-BASED INTERPRETATION OF HAMILTONIAN LEARNING PROPOSITION 2. SECOND-ORDER WEIGHTS EVOLUTION 2. SECOND-ORDER WEIGHTS EVOLUTION 2. SECOND-ORDER WEIGHTS EVOLUT
PERSONAL PROPOSITION 2. SECOND-ORDER WEIGHTS EVOLUTION 2. SECOND-ORDER WEIGHTS ENDING 2. SECOND-ORDER WEIGHTS *The Hamiltonian evolution dictated by Eqs. 10 leads to the following second-*

$$
\ddot{w}_{ij} + \theta_{ij}\dot{w}_{ij} + \beta_{ij}\dot{p}_{ij} = 0
$$
\n
$$
\dot{p}_{ij} = -\alpha_i \omega_{ij} \sigma'(a_i) p_i \xi_j
$$
\n
$$
\int_{g_{ij}} g_{ij}
$$

Eq. 10 - Eq. 10 - Eq. 16 gives a straightforward expression for the evolution of the e II - gradient-based interpretation

S3P-2024 - Neural propagation $10₁$ $10₂$ $10₃$ 10

S3P-2024 - Neural propagation Theorem 1. **1.** $\frac{1}{2}$ **1.** $\frac{1}{2}$

ENERGY BALANCE (con't) $T = \text{NFRGY RAI}$ ΔNCF (con't) even though the role of *ij* is replaced with *ij* .

Theorem 1. *-* I Principle of Cognidynamics **The system of the system of the system of the system of the energy balance system of the energy balance I Principle of Cognidynamics**

j i (*x*¹ C₁ IC₁ (*x*₂ cC₁ III Carl CILIICI DC pOSILIVC All energy term can either be positive or negative!

> S3P-2024 - Neural propagation 024 - Neural propaga

Proof.

$$
H_s|_{s=\tau} = \frac{\partial}{\partial s} \Big(-\frac{1}{2} \sum_{ij} \beta_{ij}(s) p_{ij}^2(s) + \sum_{i \in \mathcal{O}} \phi_i(s) V(\xi(s), s) + \sum_{i \in \bar{\mathcal{V}}} \alpha_i(s) p_i(s) \Big[-\xi_i(s) + \sigma \Big(\sum_j \omega_{ij}(s) w_{ij}(s) \xi_j(s) \Big) \Big] \Big) \Big|_{s=\tau} = -\sum_{ij} \dot{\beta}_{ij}(\tau) p_{ij}^2(\tau) + \sum_{i \in \mathcal{O}} \phi_i(\tau) V_s(\xi_i(\tau), \tau) + \dot{\phi}_i(\tau) V(\xi_i(\tau), \tau) + \sum_{i \in \bar{\mathcal{V}}} \dot{\alpha}_i(\tau) p_i(\tau) \Big[-\xi_i(\tau) + \sigma \Big(\sum_j \omega_{ij}(\tau) w_{ij}(\tau) \xi_j(\tau) \Big) \Big] + \sum_{i \in \bar{\mathcal{V}}} \alpha_i(\tau) p_i(\tau) \sigma'(a_i(\tau)) \sum_j \dot{\omega}_{ij}(\tau) w_{ij}(\tau) \xi_j(\tau)
$$

Now, let $\Delta H := H(\xi(t), w(t), p_{\xi}(t), p_w(t), t) - H(\xi(0), w(0), p_{\xi}(0), p_w(0), 0)$ be. If we integrate over $[0, t]$ we get

$$
\Delta H = \int_0^t \sum_{i \in \mathcal{O}} (\phi_i(\tau) V_s(\xi_i(\tau), \tau)) d\tau \leftarrow E
$$

+
$$
\int_0^t \dot{\phi}_i(\tau) V(\xi_i(\tau), \tau) d\tau \leftarrow -D_\phi
$$

-
$$
\frac{1}{2} \int_0^t \sum_{i,j} \dot{\beta}_{ij}(\tau) p_{ij}^2(\tau) d\tau \leftarrow -D_\beta
$$

+
$$
\int_0^t \sum_{i \in \bar{\mathcal{V}}} \dot{\alpha}_i(\tau) p_i(\tau) \left[-\xi_i(\tau) + \sigma(a_i(\tau)) \right] d\tau \leftarrow -D_\alpha
$$

+
$$
\int_0^t \sum_{i \in \bar{\mathcal{V}}} \alpha_i(\tau) p_i(\tau) \sigma'(a_i(\tau)) \sum_j \dot{\omega}_{ij}(\tau) w_{ij}(\tau) \xi_j(\tau) d\tau \leftarrow -D_\omega
$$

S3P-2024 - Neural propagation

DEVELOPMENTAL LEARNING AND ENERGY-DRIVEN HEURISTICS

CONSCIOUSNESS ISSUES the conscious modify the conscious modify the Lagrangian function, the optimization, the optimi

S3P-2024 - Neural propagation by a higher level of abstraction by a strategies of abstraction by a strategies

NEURAL VS ELECTROMAGNETIC WAVE PROPAGATION

WAVE PROPAGATION

MAXWELL'S EQUATIONS A VIA/FLUC FOUATION C obtained by approximating each derivative (whether in space or time) by

... plus divergence equations

S3P-2024 - Neural vs wave propagation scalar one-way wave equation u^t +u^x = 0. Centered approximations in space

MAXWELL EQS: INVERSE PROBLEM PIANTELL EQS: INT MAXWELL EOS: INVERSE PROBLEM DECAMELL FOC. IN UEDET PROPLEM where *D*curl represents the discretized curl operator, and *J*(*t*) is the current density (external source). ⇢

$$
\begin{cases}\n\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{(Gauss's Law)} \\
\nabla \cdot \mathbf{B} = 0 \quad \text{(Gauss's Law for Magnetic)} \\
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday's Law)} \\
\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{(Ampère's Law with Maxwell's correction)} \\
D_{\text{div}} E(t) = \frac{\rho(t)}{\epsilon_0}, \quad D_{\text{div}} B(t) = 0 \\
\frac{d}{dt} \begin{pmatrix} E \\ B \end{pmatrix} (t) = \begin{pmatrix} 0 & -D_{\text{curl}} \\ D_{\text{curl}} & 0 \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix} (t) + \frac{1}{\epsilon_0} \begin{pmatrix} S \\ 0 \end{pmatrix} J(t) \\
\dot{x}(t) = \mathbf{A} x(t) + \mathbf{B} \overline{u(t)} \qquad y(t) := C \begin{pmatrix} E \\ B \end{pmatrix} (t) = C x(t)\n\end{cases}
$$

Do: \angle *t* S3P-2024 - Neural vs wave propagation S3P-2024 - Neural vs wave propagation

INVERSE PROBLEM AS OPTIMAL CONTROL

where *C* is the measurement matrix selecting specific spatial points. We formulate the ISP as the divergence equations on B, E

$$
u^* = \arg\min_u \int_0^\infty \left[(Cx(t) - z(t))'Q(Cx(t) - z(t)) + u(t)'Ru(t) \right] dt
$$

\n
$$
A'P + PA - PBR^{-1}B'P + C'QC = 0
$$
 It's likely very hard to solve!
\n
$$
u(t) = -R^{-1}B'Px(t)
$$

This minimizes the cost function and ensures the system tracks the target *z*(*t*). It is worth mentioning

33 P-2024 - INeural VS W S3P-2024 - Neural vs wave propagation

HAMILTONIAN SOLUTION HAMILTONIAN SOLUTION

$$
S = BR^{-1}B'
$$

$$
H(x, p) = \frac{1}{2}(Cx(t) - z(t))'Q(Cx(t) - z(t)) - \frac{1}{2}p'Sp
$$

$$
\begin{aligned}\n\dot{x}(t) &= Ax(t) - Sp(t) & \dot{x}(t) &= Ax(t) + Bu(t) \\
\dot{p}(t) &= -C'Q(Cx(t) - z(t)) - A'p(t) & \text{generally hard to solve} \\
x(0) &= x_0 \\
p(0) &= p_0\n\end{aligned}
$$

 $B'p(t)$ $\mathcal{L}(t) = \mathbf{D}^{-1} \mathbf{D}' \mathbf{u}(t)$ $u(t) = -R^{-1}B'p(t)$

S3P-2024 - Neural vs wave propagation

TME SYMMETRY *H*(*x, u, p*) (3)

 $\lim_{h \to 0}$ = symmetry . Hamilton's equations are as follows: \mathbb{R}^n

> *^H*(*x, p*) = ¹ A eigenvalues on the imaginary axis *x*˙(*t*) = *Ax*(*t*) *Sp*(*t*) *p*(*t*) (*t*) (*t*

 $\dot{x}(t) = Ax(t) - Sp(t)$ $\dot{x}(t) = Ax(t) + Bu(t)$ $\dot{p}(t) = -C'Q(Cx(t) - z(t)) - A'$ $p(t)$ $x(0)$ in x_0 \overrightarrow{p} boundary conditions \mathcal{L} **x**(0) $\frac{1}{2}$ $\frac{b'}{c}$ Now let us consider any interval [0*, T*] with *T <* 1. Since Maxwell's equations are reversible, if "forward propagation also for the co-state!" $\begin{array}{ccc} \cdot\end{array}$ $\begin{array}{ccc} \cdot\end{array}$ in initial $\begin{array}{ccc} \cdot\end{array}$ in $\begin{array}{ccc} \cdot\end{array}$ $u(t) = A u(t) + I(t)$ $x(0) \leq x_0$ *p*(0) = *p*⁰ $u(t) = -R^{-1}B'p(t)$ $\alpha_{\rm g}(1)$ $Cx(t)-z(t))-A'p(t)$ $x(0) \leq x_0$ solve the ODE for \mathbb{R}^n \mathcal{F}_{max} *R*¹*B*⁰ $\frac{d}{dx}$ $\frac{d}{dx}$ $\frac{d}{dx}$ $\frac{d}{dx}$ $\frac{d}{dx}$ No, *Dcurl* is antisymmetric and, therefore, also *A* = *A*⁰ and, since *B* and *R >* 0 are symmetric we $\dot{u}(t) = A'u(t) + R^{-1}B'C'Q(Cx(t) - z(t))$ $\big)$

S3P-2024 - Neural vs wave propagation Theorem 1. *The LQ formulation of ISP corresponds with inverting Maxwell's equations 2 into the* S3P-2024 - Neural vs wave propagation

S3P-2024 - Neural vs wave propagation

Stimulation of fresh ideas CONCLUSIONS

- Regulated access to data collections and the challenge of CollectionLess AI - emphasis on environmental interactions
- Learning theory inspired from Theoretical Physics; a prealgorithmic step: Cognitive Action, natural laws vs algorithms)
- Hamiltonian Learning and dissipation
- Local SpatioTemporal Propagation (LSTP) as a proposal to replace Backpropagation in "temporal learning environments"
- Electromagnetic wave propagation

Stimulation Constitution HIRING AT SAILAB on Collectionless AI

Two postdoc positions

2 years (50 KEuro/year)